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Finite-rank constraints on linear flows and the Davey–Stewartson equation

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Abstract. Finite-rank constraints for the right-derivatives of certain automorphisms solving the heat equation imply the Davey–Stewartson system. Solutions comprising Wronskian and Grammian determinants are found and the real reductions for the Davey–Stewartson I and Davey–Stewartson II are considered.

1. Introduction

In this work we develop a method for the construction of solutions of the two-component Kadomtsev–Petviashvili (KP) or Davey–Stewartson (DS) equation. We shall also treat its two real reductions DSI and DSII, in their focusing and defocusing cases which correspond to elliptic and hyperbolic second-order differential operators respectively. Generalizations to the multicomponent case are straightforward.

The solutions in which we are interested can be described functionally in terms of the heat equation in one dimension with complex variables, which is replaced by the Schrödinger equation when dealing with the real reductions. Some of the solutions were obtained, apparently by direct computation in Freeman *et al* (1990), Hietarinta and Hirota (1990) and Nimmo (1992), as a 'double' Wronskian for the DS system—those in Hietarinta and Hirota (1990) were reduced to the dromion solutions of the DSI equation—and in Gilson and Nimmo (1991) as a 'double' Grammian for the DSI equation, formed with solutions of the Schrödinger equation. In Degasperis (1990) the spectral analysis of Fokas and Santini (1989, 1990) and Santini (1990) was used to obtain, for the first time, the 'double' Grammian solutions of DSI.

In Guil *et al* (1994) we were able to construct the Wronskian expressions for solutions of the KP equation through a dimensional reduction of a KP equation in an infinite dimensional space. The procedure described there reproduces in the infinite dimensional case the well known connection between the heat and Burgers equations. The same Hopf–Cole transformation puts, as in the Burgers case, the heat hierarchy in the group into the KP hierarchy in the corresponding Lie algebra. To obtain the scalar KP equation only, one selects the appropriate initial conditions. Besides Wronskians we have derived the solutions of the KP given by Grammian determinants in Mikae *et al* (1990), Nakamura (1989) and Zakharov and Shabat (1974) from this formulation.

It is the purpose of this work to reproduce for the DS equation the construction given for the KP equation. To solve the easy part of the problem one takes as the starting point the

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two-dimensional heat equation (or n-dimensional if one wants the n-component KP) subject to certain initial conditions (see equations (3.7)) which yield the desired DS system. The difficulty in reducing this to the DSI or DSII equations appears in selecting the correct initial conditions for these real cases. We have solved the problem for the solutions mentioned before, those given as 'double' Wronskians, 'double' Grammians or a combination of them (henceforth simply Wronskians and Grammians). The Wronskian–Grammian solutions for the DSII equation obtained in this paper are, to the authors' knowledge, new.

The organization and contents of the paper are as follows. In section 2 we describe the geometric origin of the initial conditions for the n-component KP equation, which is ultimately related to a problem of separation of variables. The connection of the DS system with the heat equation is given in section 3 where we obtain general formulae expressing its solutions as determinants or quotients of them. We derive, in section 4, solutions in the form of Grammian and Wronskian determinants and present finally, in section 5, an effective procedure to reduce these solutions to the DSI and DSII equations.

2. Splitting the zero-curvature condition: the Darboux coefficients

The integrable nonlinear equations we shall consider admit a simple description in terms of the theory of linear flows in linear spaces V.

For a collection of operator-valued functions $M_i(x_1, \ldots, x_n) \in L(V)$ depending on the n complex variables $\{x_i\}_{i=1,\ldots,n}$ we define the linear flows

$$\frac{\partial \Psi}{\partial x_i} = M_i \Psi \qquad i = 1, \ldots, n$$

where $\Psi(x_1, \ldots, x_n) \in V$. The compatibility conditions among them result in the zero-curvature equations

$$\partial_i M_j - \partial_j M_i + M_i M_j - M_j M_i = 0 \tag{2.1}$$

where $\partial_i = \partial/\partial x_i$, which are a set of nonlinear partial-differential equations for the operators M_i. On the other hand, given a family of linear operators M_i satisfying equations (2.1) one knows that locally there exists an automorphism $\psi(x_1, \ldots, x_n) \in GL(V)$ such that

$$M_i = \partial_i \psi \cdot \psi^{-1}. \tag{2.2}$$

Equations (2.1) can be written as

$$\partial_i M_j + M_j M_i = \partial_j M_i + M_i M_j$$

from which we obtain a particular solution of (2.1) by imposing

$$\partial_i M_j + M_j M_i = 0 \qquad i \neq j. \tag{2.3}$$

From equations (2.3) one deduces the compatibility conditions

$$(\partial_i \partial_j - \partial_j \partial_i) M_k = 0$$

and consequently equations (2.3) form a closed system.

For the automorphism ψ we have

Proposition 1. The collection of operators $M_i = \partial_i \psi \cdot \psi^{-1}$, $i = 1, \ldots, n$, $\psi(x_1, \ldots, x_n) \in GL(V)$, solves equation (2.3) if and only if

$$\psi(x_1, \ldots, x_n) = \psi_1(x_1) + \cdots + \psi_n(x_n). \tag{2.4}$$

Proof. Introducing the expression (2.2) for M_i in equation (2.3) one concludes

$$\partial_i \partial_j \psi = 0 \qquad i \neq j$$

the solution of which is given in the proposition. □

Thus, system (2.3) represents in $L(V)$ the process of separation of variables in the group $GL(V)$. To proceed further we consider a linear splitting of V,

$$V = V_1 \oplus \cdots \oplus V_n$$

and we require that $\operatorname{im} M_i(x_1, \ldots, x_n) \subset V_i$. Associated with this decomposition one has the representation

$$M_i = \sum_{j=1}^{n} M_{ij}$$

where

$$M_{ij} = M_i|_{V_j} : V_j \to V_i.$$

Now, we can separate (2.3) into two subsystems, namely

$$\partial_i M_{ji} + M_{ji} M_{ii} = 0 \qquad i \neq j$$

and

$$\begin{cases} \partial_i M_{jk} + M_{ji} M_{ik} = 0 & i \neq j, i \neq k, j \neq k \\ \partial_i M_{jj} + M_{ji} M_{ij} = 0 & i \neq j. \end{cases} \tag{2.5}$$

Observe that equations (2.5) are invariant under the transformation $M_{ii} \to A_i + M_{ii}$, $A_i \in L(V_i)$. Notice also that the ψ_is can be taken with their images in V_i.

An interesting reduction of this system is obtained once the M_i, modulo the invariance cited in the above paragraph, are requested to belong to the left ideal of finite rank operators. The simplest case is rank one, if we let $e_i \in V_i$ be the vector expanding the one-dimensional image, we obtain the representation

$$M_i = A_i + e_i \otimes \alpha_i \tag{2.6}$$

where $\alpha_i(x_1, \ldots, x_n) \in V^*$ is a linear functional over V. For the functions

$$p_{ij} := \langle \alpha_i, e_j \rangle \tag{2.7}$$

we have the following proposition.

Proposition 2. Let p_{ij} be the functions defined in (2.7), then if $\{M_i\}$ as given by equation (2.6) is a solution of (2.3) then

$$\partial_k p_{ij} + p_{ik} p_{kj} = 0 \qquad i, j \neq k. \tag{2.8}$$

Proof. From equations (2.3) and (2.6) it follows that

$$\partial_k \alpha_i + \alpha_i A_k + p_{ik} \alpha_k = 0$$

its contraction with e_j gives the desired result. □

The functions p_{ij} satisfy the equations defining the Darboux coefficients or rotation coefficients for an n-orthogonal system of curvilinear coordinates. Equations (2.8) are also the compatibility conditions for the linear system

$$\partial_j h_i + p_{ij} h_j = 0.$$

The coefficients h_i give the diagonal metric of zero curvature $\mathrm{d}s^2 = \sum_i \sigma_i h_i^2 \, \mathrm{d}x_i^2$, $\sigma_i \in \{\pm 1\}$.

Equations (2.8) constitute a non-trivial system of equations obtained from a trivial one, namely equations (2.3). The non-trivial task is to find an automorphism, as in equation (2.4), with its right-derivatives constrained by the expression given in (2.6).

3. The Davey–Stewartson system: construction of solutions

An interesting class of equations connected to equations (2.8) is represented by the symmetries associated with the Laplacian.

In this section we take $n = 2$, thus we have only two coordinates x_1, x_2, a splitting $V = V_1 \oplus V_2$, and A_1 (A_2) a linear operator over V_1 (V_2). The functions $p_{11}, p_{22}, p_{12}, p_{21}$ of (2.7) satisfy

$$\partial_1 p_{22} = \partial_2 p_{11} = -p_{12}p_{21}$$

therefore p_{ii} admits a potential function, say U, such that

$$p_{ii} = \partial_i U \tag{3.1}$$

so that

$$\partial_1\partial_2 U + p_{12}p_{21} = 0.$$

We can add an additional time parameter t by considering the symmetry

$$\partial_t \psi = \Delta\psi \tag{3.2}$$

where $\Delta := \partial_1^2 + \partial_2^2$ is the two-dimensional Laplacian. That is, ψ is a $GL(V)$ valued solution of the two-dimensional heat equation, its right-derivatives being solutions of

$$\partial_t M_i = \Delta M_i + 2\sum_{j=1}^{2}(\partial_j M_i)M_j. \tag{3.3}$$

Now we show that this system is connected with the DS system.

Theorem 1. Let $\psi(x_1, x_2, t)$ be a solution of equation (3.2) with its right-derivatives M_1, M_2 as in (2.6). Then, the functions p_{12}, p_{21} and U defined in (2.7) and (3.1) satisfy

$$\partial_1\partial_2 U + p_{12}p_{21} = 0 \tag{3.4}$$

$$\partial_t p_{12} - (\partial_1^2 - \partial_2^2)p_{12} - 2p_{12}(\partial_1^2 - \partial_2^2)U = 0 \tag{3.5}$$

$$\partial_t p_{21} + (\partial_1^2 - \partial_2^2)p_{21} + 2p_{21}(\partial_1^2 - \partial_2^2)U = 0. \tag{3.6}$$

Proof. Taking into account the expression (2.6) from (3.3) one finds

$$\partial_t\alpha_i = \Delta\alpha_i + 2\sum_{j=1}^{2}((\partial_j p_{ij})\alpha_j + (\partial_j\alpha_i)A_j) \qquad i = 1, 2$$

that, upon contraction with e_j and with the definition of the potential U, gives the desired result. □

We now proceed to construct the general solution ψ to equation (3.2) and with right-derivatives as in (2.6). For this aim the following observation is particularly useful. (Just use $\alpha_i\psi = \beta_i$.)

Proposition 3. The system of equations (3.2), (2.2) and (2.6) is equivalent to

$$\begin{aligned} \partial_i\psi &= A_i\psi + e_i \otimes \beta_i \qquad i = 1, 2 \\ \partial_t\psi &= \Delta\psi \end{aligned} \tag{3.7}$$

where $\beta_i(x_i, t) \in V^*$ depends only on the coordinate x_i and satisfies

$$\partial_t\beta_i(x_i, t) = \partial_i^2\beta_i(x_i, t). \tag{3.8}$$

Now, define

$$\psi_0(x_1, x_2, t) := \exp\left[A_1 x_1 + A_2 x_2 + (A_1^2 + A_2^2)t\right]$$

and

$$b_i(x_i, t) := \psi_0(x_1, x_2, t)^{-1} e_i \in V_i.$$

Then we obtain the following proposition.

Proposition 4. The solution to equations (3.2), (2.2) and (2.6) is given by

$$\psi = \psi_0 \cdot \varphi$$

with an invertible operator

$$\varphi(x_1, x_2, t) = C + \partial_1^{-1}(b_1(x_1, t) \otimes \beta_1(x_1, t)) + \partial_2^{-1}(b_2(x_2, t) \otimes \beta_2(x_2, t)) \tag{3.9}$$

where ∂_i^{-1} is a primitive commuting with ∂_i, and $C \in L(V)$.

Proof. Equations (3.7) can be written as

$$(\partial_i - A_i)\psi_i = e_i \otimes \beta_i$$

where we have used equation (2.4). Therefore,

$$\psi_i = \psi_{0,i} C_i + (\partial_i - A_i)^{-1}(e_i \otimes \beta_i) \tag{3.10}$$

with $\psi_{0,i} = \exp[A_i x_i + A_i^2 t]$, $C_i : V \to V_i$ and $(\partial_i - A_i)^{-1}$ stands for the operator $(\partial_i - A_i)^{-1} = \psi_{0,i}\partial_i^{-1}\psi_{0,i}^{-1}$, we set $C = C_1 + C_2$ and $\psi = \psi_1 + \psi_2$ to get the desired result. □

We define

$$\xi_{ij} := \psi + e_j \otimes \beta_i - e_j \otimes \eta_j \psi$$
$$\zeta_{ij} := \varphi + b_j \otimes \beta_i - b_j \otimes \delta_j \varphi$$

where $\eta_j, \delta_j(x_1, x_2, t) \in V^*$ are such that

$$\langle \eta_j, e_j \rangle = \langle \delta_j, b_j \rangle = 1 \qquad j = 1, 2.$$

Note that for a given η_j we can take $\delta_j = \eta_j \psi_0$ and, if this is the case, then $\xi_{ij} = \psi_0 \zeta_{ij}$.

Now, we can state the main result of this paper.

Theorem 2. If φ, as given by equation (3.9), has a determinant then

$$U = \ln \det \varphi \tag{3.11}$$

$$p_{ij} = \langle \beta_i, \varphi^{-1} b_j \rangle = \frac{\det \xi_{ij}}{\det \psi} = \frac{\det \zeta_{ij}}{\det \varphi} \tag{3.12}$$

are a solution of equations (3.4), (3.5) and (3.6).

Proof. From the relation $\alpha_i\psi = \beta_i$ one concludes

$$e_i \otimes \alpha_i = \psi_0(\partial_i\varphi \cdot \varphi^{-1})\psi_0^{-1}$$

and hence, if φ has a determinant,

$$\partial_i U = p_{ii} = \mathrm{Tr}(\partial_i\varphi \cdot \varphi^{-1}).$$

Then, up to an additive constant, we obtain for U the expression

$$U = \ln\det\varphi.$$

Since

$$\xi_{ij} = [1 + e_j \otimes (\alpha_i - \eta_j)]\psi$$
$$\zeta_{ij} = [1 + b_j \otimes (\beta_i\varphi^{-1} - \delta_j)]\varphi$$

their determinants are

$$\det\xi_{ij} = \det[1 + e_j \otimes (\alpha_i - \eta_j)]\det\psi = p_{ij}\det\psi$$
$$\det\zeta_{ij} = \det[1 + b_j \otimes (\beta_i\varphi^{-1} - \delta_j)]\det\varphi = p_{ij}\det\varphi$$

from whence the expressions for p_{12} and p_{21} follows. □

Notice that in equation (3.12) the computation of the inverse of φ is avoided by the use of determinants. This is an advantage of determinant-type expressions.

4. Particular solutions of the DS system

Concrete choices for V, A_i, e_i and β_i give different sets of solutions to the DS system. In this section we will be concerned about two particular families of solutions. First, we extend the Grammian determinant-type formulae from Mikae *et al* (1990), Nakamura (1989) and Zakharov and Shabat (1974) for solutions of the KP equation to the DS case, and obtain the same generalization for the solutions of the KP constructed in Chau *et al* (1992) encompassing Wronskians and Grammian determinants. The Wronskian solutions for the DS system, that are contained as examples in the formulae given below, were obtained in Freeman *et al* (1990) and Nimmo (1992). For the KP case see Freeman and Nimmo (1983) and Satsuma (1979).

Theorem 3. Denote by W_i, $i = 1, 2$, two linear spaces, let $s_i(x_i, t) \in W_i$, $i = 1, 2$, be solutions of

$$\partial_t s_i = -\partial_i^2 s_i$$

choose $\varsigma_i(x_i, t) \in (W_1 \oplus W_2)^*$ such that $\langle\varsigma_i, s_i\rangle = 1$ and let $\sigma_i(x_i, t) \in (W_1 \oplus W_2)^*$, $i = 1, 2$, be solutions of

$$\partial_t\sigma_i = \partial_i^2\sigma_i.$$

For any $C \in L(W_1 \oplus W_2)$ we define

$$\Phi := C + \partial_1^{-1}(s_1 \otimes \sigma_1) + \partial_2^{-1}(s_2 \otimes \sigma_2)$$

and

$$\Phi_{ij} := \Phi + s_j \otimes (\sigma_i - \varsigma_j\Phi).$$

Then,

$$U = \ln\det\Phi$$
$$p_{ij} = \frac{\det\Phi_{ij}}{\det\Phi}$$

is a solution of equations (3.4), (3.5) and (3.6).

Proof. As V_i, $i = 1, 2$, we take the space of W_i-valued sequences

$$V_i = \ell_{W_i} \cong W_i \oplus W_i \oplus \cdots.$$

An element in V_i is of the form $(a_n)_{n\geqslant 0}$ with $a_n \in W_i$. The shift operator $\Lambda_i \in L(V_i)$ is defined as

$$\Lambda_i (a_n)_{n\geqslant 0} = (a_{n+1})_{n\geqslant 0}$$

and we take

$$A_i = \Lambda_i \qquad i = 1, 2.$$

This choice implies that

$$b_i = (s_i, -\partial_i s_i, \partial_i^2 s_i, \ldots)$$

where s_i takes values in W_i. This follows from the equation $\partial_i b_i = -A_i b_i$. The evolution of b_i: $\partial_t b_i = -\partial_i^2 b_i$ implies the same equation for s_i.

For the β_i's our choice is

$$\beta_i((a_n)_{n\geqslant 0} \oplus (b_n)_{n\geqslant 0}) = \langle \sigma_{i1}, a_0\rangle + \langle \sigma_{i2}, b_0\rangle$$

where σ_{ij} takes its values in W_j^* and, because β_i satisfies (3.8), so does $\sigma_i = \sigma_{i1} \oplus \sigma_{i2}$ that takes values in $(W_1 \oplus W_2)^*$. A proper C can be taken such that

$$\det \varphi = \det \Phi$$
$$\det \zeta_{ij} = \det \Phi_{ij}$$

and from theorem 2 we finally obtain the desired result. Here δ_j must be chosen in a similar way as that used for the β_i, replacing σ_i by ς_i. □

In the finite-dimensional case, $\dim W_i = N_i$, if we fix two bases $\mathcal{B}_i$ in W_i for $i = 1, 2$ respectively, we can complete the above theorem as follows. Let $(s_i)_n$, $n = 0, \ldots, N_i - 1$ be the components of s_i in the basis $\mathcal{B}_i$. Define $\Phi_1^n, n = 0, \ldots, N_2 - 1$ ($\Phi_2^n, n = 0, \ldots, N_1 - 1$) as the matrix obtained from that corresponding to Φ by replacing the $(N_1 + n - 1)$th (nth) row by σ_1 (σ_2), then:

Proposition 5. For the p_{ij} of theorem 3 one has the expressions

$$p_{ij} = \frac{\sum_{n=0}^{N_j - 1} (s_j)_n \det \Phi_i^n}{\det \Phi}.$$

Proof. Use the first formula of equation (3.12). □

To include Wronskian-type expressions in the formulae stated previously we need the following observation.

Proposition 6. The operator ψ can be expressed as

$$\psi = \sum_{i=1,2} \left[\sum_{n=0}^{M_i - 1} e_i^n \otimes \partial_i^n \gamma_i + \psi_0 (C_i + \partial_i^{-1}(g_i \otimes \gamma_i)) \right]$$

where

$$e_i^n = A_i^{M_i - 1 - n} e_i \qquad i = 1, 2$$

and

$$\beta_i = \partial_i^{M_i} \gamma_i \qquad g_i = A_i^{M_i} b_i = (-\partial_i)^{M_i} b_i.$$

Proof. From equation (3.10) we formally deduce

$$\psi_i = \psi_{0,i} C_i + (1 - A_i \partial_i^{-1})^{-1} (e_i \otimes \partial_i^{-1} \beta_i)$$

which can be written as

$$\psi_i = \psi_{0,i} C_i + \sum_{n \geqslant 0} A_i^n e_i \otimes \partial_i^{-n-1} \beta_i .$$

For any couple M_1, M_2 of natural numbers we have

$$\psi_i = \sum_{n=0}^{M_i - 1} e_i^n \otimes \partial_i^n \gamma_i + \psi_{0,i} C_i + (\partial_i - A_i)^{-1} (e_i^{M_i} \otimes \gamma_i) \qquad i = 1, 2.$$

Now, the methods in the proof of proposition 4 can be applied to obtain the desired result. □

With this proposition at hand we are able to formulate the following theorem.

Theorem 4. Let $s_i(x_i, t)$ be as in theorem 3 and $\overline{W}_i$, $i = 1, 2$, an M_i-dimensional linear space with basis $\{e_i^n\}_{n=0}^{M_i - 1}$. *Consider* $\sigma_i(x_i, t) \in (\overline{W}_1 \oplus W_1 \oplus \overline{W}_2 \oplus W_2)^*$ *as in theorem 3* and introduce the map

$$\mathcal{W} : \overline{W}_1 \oplus W_1 \oplus \overline{W}_2 \oplus W_2 \to W_1 \oplus W_2$$

given by

$$\mathcal{W} := \sum_{\substack{n=0,\ldots,M_i-1 \\ i=1,2}} e_i^n \otimes \partial_i^n \sigma_i .$$

Now, we define

$$\Phi := \mathcal{W} + C + \partial_1^{-1}(s_1 \otimes \sigma_1) + \partial_2^{-1}(s_2 \otimes \sigma_2)$$

and

$$\Phi_{ij} := \Phi - e_j^{M_j - 1} \otimes (\partial_j^{M_j - 1} \sigma_j - \partial_i^{M_i} \sigma_i).$$

Then,

$$U = \ln \det \Phi$$
$$p_{ij} = \frac{\det \Phi_{ij}}{\det \Phi}$$

is a solution of equations (3.4), (3.5) and (3.6).

Proof. For the linear space V_i we take

$$V_i = \overline{W}_i \oplus \ell_{W_i}$$

where $\overline{W}_i$ is an M_i-dimensional linear space with basis $\{e_i^n\}_{n=0}^{M_i - 1}$. The operators A_i are defined by

$$A_i(\overline{w} \oplus (a_n)_{n \geqslant 0}) = \overline{\Lambda}_i \overline{w} \oplus (\Lambda_i (a_n)_{n \geqslant 0} + \overline{w}_0 u_i)$$

where $\overline{w} = \sum_{n=0}^{M_i - 1} \overline{w}_n e_i^n$, $\overline{\Lambda}_i$ is the shift operator in $\overline{W}_i$ and $u_i \in \ell_{W_i}$ is a constant sequence. Observe that $u_i = A_i e_i^0$ and that A_i acts on ℓ_{W_i}. Thus $g_i(0) = A_i^{N_i} e_i = u_i \in \ell_{W_i}$ and $g_i(x_i, t) \in \ell_{W_i}$. Therefore, applying similar arguments as those used in the proof of theorem 3 we conclude that

$$g_i = 0 \oplus (s_i, -\partial_i s_i, \partial_i^2 s_i, \ldots).$$

For the γ_i, we take $P|_{(\ell_{W_i})^*} \gamma_i$ that has as kernel those sequences in which the first component is identically zero. An appropriate choice for C is now enough to conclude the truth of the theorem (with the aid of theorem 2 and proposition 6). □

There is an intermediate case between the two theorems in this section. Observe that in the theorem above M_1, M_2 are natural numbers, and therefore different from zero. The case when both are zero is considered in theorem 3. If $M_1 > 0$ but $M_2 = 0$ then for p_{21} we use the expressions in theorem 4 ($\gamma_2 = \beta_2$) and for p_{12} that of theorem 3 (with $\beta_1 = \partial_1^{M_1}\gamma_1$).

5. The reductions to DSI and DSII

In this section we modify our symmetry to

$$\mathrm{i}\partial_t\psi = (\partial_1^2 - \partial_2^2)\psi.$$

So that the DS system reads

$$\begin{aligned}
&\partial_1\partial_2 U + p_{12}p_{21} = 0\\
&\mathrm{i}\partial_t p_{12} - \Delta p_{12} - 2p_{12}\Delta U = 0\\
&\mathrm{i}\partial_t p_{21} + \Delta p_{21} + 2p_{21}\Delta U = 0.
\end{aligned}$$

Now, theorem 2 holds if β_i is a solution of

$$\mathrm{i}\partial_t\beta_1 = \partial_1^2\beta_1 \qquad -\mathrm{i}\partial_t\beta_2 = \partial_2^2\beta_2 \tag{5.1}$$

and theorems 3 and 4 need that s_i and σ_i, $i = 1, 2$, be solutions of

$$\begin{aligned}
&\mathrm{i}\partial_t s_1 = -\partial_1^2 s_1 \qquad \mathrm{i}\partial_t\sigma_1 = \partial_1^2\sigma_1\\
&\mathrm{i}\partial_t s_2 = \partial_2^2 s_2 \qquad \mathrm{i}\partial_t\sigma_2 = -\partial_2^2\sigma_2.
\end{aligned}$$

5.1. The DSI case

The DSI reduction appears when $x_1 = \xi$, $x_2 = \eta \in \mathbb{R}$ and

$$p_{12} = \varepsilon p_{21}^* =: p \qquad \varepsilon = \pm 1 \qquad \nabla U(\xi, \eta, t) \in \mathbb{R}^2.$$

This implies the differential equations

$$\partial_\xi\partial_\eta U + \varepsilon|p|^2 = 0 \tag{5.2}$$

$$\mathrm{i}\partial_t p - \Delta p - 2p\Delta U = 0. \tag{5.3}$$

These equations are just the DSI in its defocusing, $\varepsilon = 1$, and focusing, $\varepsilon = -1$, cases.

The problem to tackle here is which data A_i, β_i are suitable for this reduction. If the complex linear spaces V_1 and V_2 are furnished with scalar products and $^\dagger : V_i \to V_i^*$ denotes the standard isomorphism generated by these scalar products, a possible solution to this question is as follows.

Proposition 7. If

$$\beta_1 = b_1^\dagger H \qquad \beta_2 = \varepsilon b_2^\dagger H \tag{5.4}$$

$$C = 1 \qquad H^\dagger = H \tag{5.5}$$

then the functions defined in equation (2.7) satisfy

$$p_{12} = \varepsilon p_{21}^* \qquad \nabla U(\xi, \eta, t) \in \mathbb{R}^2.$$

Proof. Equations (5.1) hold by construction. On the other hand

$$\varphi = 1 + [\partial_\xi^{-1}(b_1 \otimes b_1^\dagger) + \varepsilon\partial_\eta^{-1}(b_2 \otimes b_2^\dagger)]H$$

so that $H\varphi = \varphi^\dagger H$, and $H\varphi^{-1} = (\varphi^{-1})^\dagger H$. Moreover,

$$p_{1j} = b_1^\dagger H\varphi^{-1}b_j \qquad p_{2j} = \varepsilon b_2^\dagger H\varphi^{-1}b_j$$

and hence

$$p_{12}^* = (b_1^\dagger H\varphi^{-1}b_2)^\dagger = b_2^\dagger(\varphi^{-1})^\dagger Hb_1 = b_2^\dagger H\varphi^{-1}b_1 = \varepsilon p_{21}$$

also $p_{jj}^* = p_{jj}$, thus $\nabla U(\xi, \eta, t) \in \mathbb{R}^2$. □

The results in theorem 3 reduce to the DSI by applying proposition 7 and implies:

Theorem 5. Let $s_1(\xi, t)$ ($s_2(\eta, t)$) be a function taking values in the complex linear space W_1 (W_2) solution of

$$-\mathrm{i}\partial_t s_1 = \partial_\xi^2 s_1 \qquad (\mathrm{i}\partial_t s_2 = \partial_\eta^2 s_2)$$

and $H \in L(W_1 \oplus W_2)$ be a Hermitian operator. Define

$$\Phi := 1 + \left[\partial_\xi^{-1}(s_1 \otimes s_1^\dagger) + \varepsilon\partial_\eta^{-1}(s_2 \otimes s_2^\dagger)\right]H$$

and

$$\hat{\Phi} := \Phi + s_2 \otimes (s_1^\dagger H - \varsigma_2\Phi).$$

where $\varsigma_2 \in (W_1 \oplus W_2)^*$ is such that $\langle\varsigma_2, s_2\rangle = 1$.

Then,

$$U = \ln\det\Phi$$

$$p = \frac{\det\hat{\Phi}}{\det\Phi} \tag{5.6}$$

is a solution of equations (5.2) and (5.3).

Proof. The results follow immediately if we take into account the form of $b_i = (s_i, -\partial_i s_i, \ldots)$, choose adequately the Hermitian operator H and recall the contents of theorem 3 and proposition 7. □

When W_i are finite-dimensional, with $\dim W_i = N_i$, the expression in theorem 5 for p is complemented with:

Proposition 8. Fix a basis in W_i for $i = 1, 2$ and denote the components of s_2 by $(s_2)_n$. Let Φ_n *be the matrix associated with* Φ *when we replace the* $(N_1 - 1 - n)$*th row by* $s_1^\dagger H$. *Then,* the function p given by equation (5.6) can be written as

$$p = \frac{\sum_{n=0}^{N_2-1}(s_2)_n \det\Phi_n}{\det\Phi}.$$

Essentially the formula that appears in theorem 5 was obtained, by direct computation, in Degasperis (1990) and Gilson and Nimmo (1991) (see also Gilson 1992). Observe also that in Degasperis (1990) these solutions were derived for the first time within the inverse scattering technique employed in Fokas and Santini (1989, 1990) and Santini (1990), describing, in particular, the so-called gaussons. Here we have found a derivation of this result from basic principles and this allows us to extend the result by including Wronskian-type expressions. For this aim we need to introduce the Schur or heat polynomials, S_n, defined by

$$\exp(kx + k^2 t) =: \sum_{n\geqslant 0} S_n(x,t)k^n.$$

We use the notation

$$\mathfrak{S}_n(\xi,t) := S_n(-\xi, \mathrm{i}t) \qquad \mathfrak{S}_n(\eta,t) := S_n(-\eta, -\mathrm{i}t).$$

For these coefficients we have the relations $\partial_x S_n = S_{n-1}$ and $\partial_t S_n = \partial_x^2 S_n$. We obtain also the formulae

$$-\partial_\xi \mathfrak{S}_n(\xi,t) = \mathfrak{S}_{n-1}(\xi,t) \qquad -\mathrm{i}\partial_t \mathfrak{S}(\xi,t) = \partial_\xi^2 \mathfrak{S}(\xi,t) \tag{5.7}$$

$$-\partial_\eta \mathfrak{S}_n(\eta,t) = \mathfrak{S}_{n-1}(\eta,t) \qquad \mathrm{i}\partial_t \mathfrak{S}(\eta,t) = \partial_\eta^2 \mathfrak{S}(\eta,t). \tag{5.8}$$

Theorem 6. Let s_i and H be as in theorem 5, with the additional condition

$$\partial_\xi^{2M_1-n} s_1(0) = 0 \qquad n = 1,\ldots,M_1$$
$$\partial_\eta^{2M_2-n} s_2(0) = 0 \qquad n = 1,\ldots,M_2.$$

Take $\sigma_1(\xi,t) \in (\overline{W}_1 \oplus W_1)^*$ and $\sigma_2(\eta,t) \in (\overline{W}_2 \oplus W_2)^*$, solutions of equation (5.1) of the form

$$\sigma_1(\xi,t) = (\mathfrak{S}_{2M_1-1}(\xi,t),\ldots,\mathfrak{S}_{M_1}(\xi,t))^* \oplus s_1(\xi,t)^\dagger H$$
$$\sigma_2(\eta,t) = \varepsilon[(\mathfrak{S}_{2M_2-1}(\eta,t),\ldots,\mathfrak{S}_{M_2}(\eta,t))^* \oplus s_2(\eta,t)^\dagger H].$$

Here $\overline{W}_i$, $i = 1,2$, are complex linear spaces with basis $\{e_i^n\}_{n=0}^{M_i-1}$. Consider the operator $\mathcal{W}$ defined as in theorem 4, define P_W as the canonical projection $P_W : \overline{W}_1 \oplus W_1 \oplus \overline{W}_2 \oplus W_2 \to W_1 \oplus W_2$, and write

$$\Phi = \mathcal{W} + P_W + (-1)^{M_1}\partial_\xi^{-1}(\partial_\xi^{2M_1} s_1 \otimes \sigma_1) + (-1)^{M_2}\partial_\eta^{-1}(\partial_\eta^{2M_2} s_2 \otimes \sigma_2)$$
$$\hat{\Phi} = \Phi + e_2^{M_1-1} \otimes ((-\partial_\eta)^{M_2-1}\sigma_2 + (-\partial_\xi)^{M_1}\sigma_1).$$

Then

$$U = \ln\det\Phi$$
$$p = \frac{\det\hat{\Phi}}{\det\Phi}$$

is a solution of equations (5.2) and (5.3).

Proof. We need to apply proposition 7 to the results of theorem 4. Our framework is that appearing in the proof of theorem 4. We first split b_i as

$$b_i = \overline{b}_i \oplus \mathfrak{b}_i$$

with $\overline{b}_i$ taking values in $\overline{W}_i$ and $\mathfrak{b}_i$ in ℓ_{W_i}. Because of the particular structure of A_i we conclude

$$\begin{cases} \partial_\xi \overline{b}_1 = -\overline{\Lambda}_1 \overline{b}_1 \\ \partial_\eta \overline{b}_2 = -\overline{\Lambda}_2 \overline{b}_2 \\ \overline{b}_i(0) = e_i^{M_i-1} \end{cases} \tag{5.9}$$

$$\begin{cases} \partial_\xi \mathfrak{b}_1 = -(\overline{b}_1)_0 u_1 - \Lambda_1 \mathfrak{b}_1 \\ \partial_\eta \mathfrak{b}_2 = -(\overline{b}_2)_0 u_2 - \Lambda_2 \mathfrak{b}_2 \\ \mathfrak{b}_i(0) = 0. \end{cases} \tag{5.10}$$

From equations (5.9) we get

$$\widetilde{b}_t = (\mathfrak{S}_{M_t-1}, \ldots, \mathfrak{S}_1)^t.$$

When we express the sequence $u_i = (u_{i,0}, u_{i,1}, \ldots)$ in components, equations (5.10) give

$$\mathfrak{b}_{1,n} = \begin{cases} -\sum_{m=1}^{n} \mathfrak{S}_{M_1-m} u_{1,n-m} + (-\partial_\xi)^n \mathfrak{b}_{1,0} & n = 0, \ldots, M_1 - 1 \\ -\sum_{m=1}^{M_1} \mathfrak{S}_{M_1-m} u_{1,n-m} + (-\partial_\xi)^n \mathfrak{b}_{1,0} & n \geqslant M_1 \end{cases}$$

$$\mathfrak{b}_{2,n} = \begin{cases} -\sum_{m=1}^{n} \mathfrak{S}_{M_2-m} u_{2,n-m} + (-\partial_\eta)^n \mathfrak{b}_{2,0} & n = 0, \ldots, M_2 - 1 \\ -\sum_{m=1}^{M_2} \mathfrak{S}_{M_2-m} u_{2,n-m} + (-\partial_\eta)^n \mathfrak{b}_{2,0} & n \geqslant M_2 \end{cases}$$

and the initial conditions

$$\begin{cases} \partial_\xi^n \mathfrak{b}_{1,0}(0) = 0 & n = 0, \ldots, M_1 - 1 \\ (-\partial_\xi)^n \mathfrak{b}_{1,0}(0) = u_{1,n-M_1} & n \geqslant M_1 \end{cases}$$

$$\begin{cases} \partial_\eta^n \mathfrak{b}_{2,0}(0) = 0 & n = 0, \ldots, M_2 - 1 \\ (-\partial_\eta)^n \mathfrak{b}_{2,0}(0) = u_{2,n-M_2} & n \geqslant M_2. \end{cases}$$

From proposition 6 it follows that

$$g_{1,0} = (-\partial_\xi)^{M_1} b_1 \qquad g_{2,0} = (-\partial_\eta)^{M_2} b_2.$$

On the other hand $\beta_1 = \partial_\xi^{M_1} \gamma_1$ and $\beta_2 = \partial_\eta^{M_2} \gamma_2$. The reduction to DSI is given by proposition 7 as $\beta_1 = b_1^\dagger H$ and $\beta_2 = \varepsilon b_2^\dagger H$. Now, taking $\mathfrak{b}_{1,0} = (\partial_\xi)^{M_1} s_1$ and $\mathfrak{b}_{2,0} = (\partial_\eta)^{M_2} s_2$, and recalling formulae (5.7) and (5.8) one can easily deduce the result stated. □

5.2. *The DSII reduction*

For the DSII case we take $x_1^* = x_2$, $\nabla U = (U_x, U_y) \in \mathbb{R}^2$ where $x_1 = z = (x + \mathrm{i}y)/\sqrt{2}$, $x, y \in \mathbb{R}$, and $\varepsilon p_{21}^* = p_{12} = p$. The equations are now

$$\Delta U + 2\varepsilon |p|^2 = 0 \tag{5.11}$$

$$\mathrm{i}\partial_t p - (\partial_x^2 p - \partial_y^2 p) - 2p(\partial_x^2 U - \partial_y^2 U) = 0. \tag{5.12}$$

Equations (5.11) and (5.12) constitute the DSII in its defocusing, $\varepsilon = 1$, and focusing, $\varepsilon = -1$, cases.

The reduction can be realized once

$$V_1 \cong V_2 \cong \mathfrak{V}$$

which will be the case in this subsection. In $\mathfrak{V} \oplus \mathfrak{V}$ one has the ε-permutation operator

$$P : \mathfrak{V} \oplus \mathfrak{V} \to \mathfrak{V} \oplus \mathfrak{V}$$
$$v_1 \oplus v_2 \mapsto P(v_1 \oplus v_2) := v_2 \oplus \varepsilon v_1$$

which allows us to formulate:

Proposition 9. If

$$e_2^* = Pe_1$$
$$A_2^* = \varepsilon P A_1 P$$
$$\beta_1^* = \beta_2 T$$

where $T \in SL(\mathfrak{V} \oplus \mathfrak{V})$ satisfies $T^{-1} = \varepsilon T^*$, and

$$C = C_1 + PC_1^* T^* \qquad C_1 \in L(\mathfrak{V} \oplus \mathfrak{V}, V_1)$$

then

$$p_{12} = \varepsilon p_{21}^*$$
$$\nabla U(x, y, t) \in \mathbb{R}^2.$$

Proof. One can check that $\psi_0^* = \varepsilon P \psi_0 P$, thus $b_2^* = Pb_1$, and $\varphi^* = \varepsilon P \varphi T$. Therefore, $\det \varphi \in \mathbb{R}$ and ∇U takes real values. On the other hand

$$p_{12}^* = \langle \beta_1^*, (\varphi^*)^{-1} b_2^* \rangle = \langle \beta_2 T, \varepsilon T^{-1} \varphi^{-1} P^{-1} P b_1 \rangle = \varepsilon p_{21}.$$

□

Now, by using theorem 3, we can give examples of solutions of the DSII equation, the proof is as in the previous ones.

Theorem 7. Let $T \in SL(W_1 \oplus W_2)$ be a linear operator over the complex linear space $W_1 \oplus W_2$ with $W_i \cong \mathfrak{W}$, $i = 1, 2$, such that $T^{-1} = \varepsilon T^*$. Let $s(z, t)$ be a W_1-valued solution of

$$-\mathrm{i}\partial_t s = \partial_z^2 s$$

and take as $\sigma(z, t) \in (\mathfrak{W} \oplus \mathfrak{W})^*$ a solution of

$$\mathrm{i}\partial_t \sigma = \partial_z^2 \sigma. \tag{5.13}$$

Choose $C \in L(\mathfrak{W} \oplus \mathfrak{W}, W_1)$ and define

$$\Phi := \phi + P\phi^* T^*$$

with

$$\phi := C + \partial_z^{-1}(s \otimes \sigma)$$

and

$$\hat{\Phi} := \Phi + Ps^* \otimes (\sigma - \varsigma \Phi)$$

with $\varsigma(z, z^*, t) \in (\mathfrak{W} \oplus \mathfrak{W})^*$ such that $\langle \varsigma, Ps^* \rangle = 1$.

Then,

$$U = \ln \det \Phi$$
$$p = \frac{\det \hat{\Phi}}{\det \Phi}$$

is a solution of equations (5.11) and (5.12).

If $\mathfrak{W}$ is finite dimensional an alternative expression for p is:

Proposition 10. Suppose that in a given basis of $\mathfrak{W}$, s has components s_n and Φ_n is the matrix associated with Φ obtained by replacing the $(N+n)$th row ($N = \dim\mathfrak{W}$) by σ, then p in theorem 7 can be expressed as

$$p = \varepsilon\frac{\sum_n s_n^* \det\Phi_n}{\det\Phi}.$$

Finally, we introduce Wronksian expressions. The proof is as in previous theorems.

Theorem 8. Take $T \in SL(\overline{W}_1 \oplus W_1 \oplus \overline{W}_2 \oplus W_2)$, such that $T^{-1} = \varepsilon T^*$, where $\overline{W}_i \cong \overline{\mathfrak{W}}$ are isomorphic complex linear spaces and W_i, $i = 1, 2$, are as in theorem 7. Choose s and C as in theorem 7 and $\sigma(z, t) \in (\overline{\mathfrak{W}} \oplus \mathfrak{W} \oplus \overline{\mathfrak{W}} \oplus \mathfrak{W})^*$ a solution of equation (5.13). Given a basis $\{e_n\}_{n=1}^M$ of $\overline{W}_1$ we define

$$\mathcal{W} := \sum_n e_n \otimes \partial_z^{n-1}\sigma$$
$$\phi := \mathcal{W} + C + \partial_z^{-1}(s \otimes \sigma)$$

and

$$\Phi := \phi + P\phi^* T^*$$
$$\hat{\Phi} := \Phi - Pe_M^* \otimes \left[\varepsilon(\partial_z^{M-1}\sigma T)^* - \partial_z^M \sigma\right].$$

Then,

$$U = \ln\det\Phi$$
$$p = \frac{\det\hat{\Phi}}{\det\Phi}$$

is a solution of equations (5.11) and (5.12).

The Wronskian expressions for solutions of DSII with $\varepsilon = 1$ obtained in Freeman *et al* (1990) are contained in theorem 8 when $\dim\mathfrak{W} = 0$ and $T = 1$. The general Wronskian–Grammian and Grammian determinant solution of the DSII equation is to the authors' knowledge entirely new. The connection between the solutions presented in Arkadiev *et al* (1989a, b) will be analysed elsewhere.

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